

ON DEFORMATION QUANTIZATION OF DIRAC STRUCTURES

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ABSTRACT. Motivated by the problem of transverse deformation quantization of foliated manifolds, we describe a quantization of Dirac structures (more precisely, of those that are formal deformations of regular ones) to stacks of algebroids in the sense of Kontsevich.

1. INTRODUCTION

This paper is, roughly speaking, about transverse deformation quantization. If N is a foliated manifold, we have the sheaf of the functions constant along the leaves. If we suppose that on N we also have a transverse Poisson structure (that is, we make the above-mentioned sheaf to a sheaf of Poisson algebras), we can locally quantize the algebras of functions constant along the leaves; the result will, in general, be just a “sheaf up to homotopy”, namely a stack of algebras. It is then an overkill to start with a transverse Poisson structure; it would be enough to have an “up to homotopy transverse Poisson structure”, and we would still get a stack after quantization. The correct notion of “up to homotopy transverse Poisson structure” is in this context the following: a formal family of Dirac structures $C(\hbar)$ on N such that the leaves of $C(0)$ are precisely the leaves of the foliation. This paper is thus about quantization of Dirac structures, more precisely, about quantization of Dirac structures that are formal deformations of regular ones.

Let thus $C(\hbar)$ be a family of Dirac structures on N , formally depending on \hbar , with the property that $C(0)$ is regular, i.e. that it gives a foliation F of N . If $M \subset N$ is a local transversal (that is, a submanifold of dimension complementary to F , intersecting the leaves transversally), the Dirac structure $C(\hbar)$ induces on M a Poisson structure of the form

$$\pi = \pi^{(1)}\hbar + \pi^{(2)}\hbar^2 + \pi^{(3)}\hbar^3 + \dots$$

These local Poisson structures are glued by $C(\hbar)$ in such a way that their quantizations form a stack on N . It is proved by a straightforward use of the formality theorem of Kontsevich; this is what we do in this paper.

The stack (sheaf of categories) that we get is a deformation of the following: for any open $U \subset N$ the objects are line bundles over U , endowed with connection along the leaves of F , with curvature equal to $C(0)$; morphisms are maps of bundles preserving the connections. Locally (i.e. for nice U ’s) this category has one object up to isomorphisms and the algebra of its endomorphisms is the algebra of functions constant along the leaves.

This paper leaves at least two open questions. First is: can one quantize Dirac structures that are not deformations of regular ones? And the second (more interesting), connected with non-commutative geometry: Instead of using the sheaf of functions constant along the leaves, one can form a global object, a non-commutative algebra that is supposed to replace functions on the possibly degenerate space of leaves. What happens with this algebra after quantization of $C(\hbar)$?

This paper is entirely based on ideas from [3]. Dirac structures were used for a deformation quantization problem, using the same techniques as here, in [4]. C^* -quantization of tori with constant Dirac structures was introduced by Tang and Weinstein in [5]; in their simple case they didn’t have to use homotopy techniques of the type that appear here, and they were able to get a global non-commutative algebra.

2. DIRAC STRUCTURES

Dirac structures were introduced by Courant [1] as a common generalization of closed 2-forms and Poisson structures. First idea is to see 2-forms on a manifold N as maps $TN \rightarrow T^*N$, bivectors as maps $T^*N \rightarrow TN$, and represent both by their graphs, which are subbundles in $(T \oplus T^*)N$. Skew-symmetry of 2-forms/bivectors makes these graphs isotropic with respect to the inner product

$$\langle (u, \alpha), (v, \beta) \rangle = \alpha(v) + \beta(u).$$

A common generalization of both bivectors and 2-forms are then maximally isotropic subbundles of $(T \oplus T^*)N$.

It then turns out that the conditions on a 2-form to be closed, and on a bivector to be Poisson, can be rephrased as a condition on the subbundle. We shall do it the standard mysterious way: one introduces so-called Courant bracket on sections of $(T \oplus T^*)N$,

$$[(u, \alpha), (v, \beta)] = [u, v] + \mathcal{L}_u \beta - i_v d\alpha,$$

and then defines *Dirac structures* to be maximally isotropic subbundles of $(T \oplus T^*)N$, closed under Courant bracket.¹

Similar to symplectic leaves of Poisson structures, any Dirac structure gives us a singular foliation on N , with closed 2-forms on its leaves (contrary to Poisson structures, these closed 2-forms don't have to be symplectic). A particular case of a Dirac structure is thus a (non-singular) foliation, with a closed 2-form on its leaves. Such Dirac structures (i.e. those, whose leaves have constant dimension) are called *regular*.

3. DIRAC STRUCTURES AS HAMILTONIAN FAMILIES OF POISSON STRUCTURES

Let $L = \Gamma(\bigwedge TM)[1]$ be the DGLA of multivector fields on a manifold M (the bracket in L is the Schouten bracket and the differential is zero). Let us take another manifold B and try to find the solutions of the MC equation in the DGLA $L \otimes \Omega(B)$.² We thus look for the sections σ of degree 2 of the vector bundle $\bigwedge TM \otimes \bigwedge T^*B = \bigwedge(TM \oplus T^*B)$ over $M \times B$, satisfying the MC equation

$$d\sigma + [\sigma, \sigma]/2 = 0.$$

Such a section can be viewed as a skew-symmetric bilinear form on $T^*M \oplus TB$, and thus as a maximal isotropic subbundle of $(T \oplus T^*)(M \times B)$. A simple computation shows that this subbundle is a Dirac structure iff σ solves the MC equation. We thus have the following:

Lemma 3.1. *A solution of the MC equation in $\Gamma(\bigwedge TM)[1] \otimes \Omega(B)$ is equivalent to a Dirac structure on $M \times B$ transversal to $TM \oplus T^*B$.*

Now suppose we replace $L = \Gamma(\bigwedge TM)[1]$ by multivectors formally depending on \hbar , as they appear in deformation quantization. That is, let the DGLA L' be given by

$$L'^{-1} = C^\infty(M)[[\hbar]],$$

$$L'^i = \hbar \Gamma(\bigwedge^{i+1} TM)[[\hbar]] \text{ for } i \geq 0.$$

We have the following version of the previous lemma:

Lemma 3.2. *A solution of the MC equation in $L' \otimes \Omega(B)$ is equivalent to a (formal) family of Dirac structures $C(\hbar)$ on $M \times B$ such that $C(0)$ is a Dirac structure with leaves $\{x\} \times B$, $x \in M$.*

In analogy with Hamiltonian families of symplectic structures, we will call solutions of the MC equation in $L' \otimes \Omega(B)$ *Hamiltonian families* of formal Poisson structures on M parametrized by B (I called them “tight families” in [4], but it's better to follow an older tradition).

¹If it makes it less mysterious – sections of $(T \oplus T^*)N$ are derivations of degree -1 of $\Omega(N)[t]$, where t is an auxiliary variable of degree 2, $dt = 0$; for two such derivations ξ_1, ξ_2 we have $[(\xi_1, \xi_2)] = [d, \xi_1, \xi_2]$ and $\langle \xi_1, \xi_2 \rangle \partial_t = [\xi_1, \xi_2]$.

²Caution: here, and in similar situations, we don't mean the algebraic tensor product, but rather its natural completion: the coefficients of the objects from $L \otimes \Omega(B)$ are allowed to be any smooth functions on $M \times B$.

4. QUANTIZATION OF HAMILTONIAN FAMILIES OF FORMAL POISSON STRUCTURES

Let us denote L'' the DGLA of polydifferential operators on M formally depending on \hbar , more precisely

$$L''^{-1} = C^\infty(M)[[\hbar]],$$

$$L''^i = \hbar PD^{i+1}(M)[[\hbar]] \text{ for } i \geq 0,$$

where PD^i denotes the space of polydifferential operators

$$\underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{i \text{ times}} \rightarrow C^\infty(M).$$

By the formality theorem, for any solution σ of the MC equation in $L' \otimes \Omega(B)$ we have a solution τ in $L'' \otimes \Omega(B)$; we call the latter a *tight family of *-products* on M parametrized by B .

Let us try to understand what tight *-product families actually are. The rest of this section is completely stolen from [3]. Let us decompose τ to bihomogeneous components $\tau = \tau^0 \oplus \tau^1 \oplus \tau^2$ (the superscript means the degree in $\Omega(B)$) to see what the MC equation means; $m : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ denotes the ordinary product of functions:

$$[\tau^0 + m, \tau^0 + m]/2 = 0,$$

i.e. $m + \tau^0$ is a family of *-products on M parametrized by B ;

$$d\tau^0 + [\tau^0 + m, \tau^1] = 0,$$

i.e. $\tau^1 \in \Omega^1(B) \otimes \hbar D(M)[[\hbar]]$, understood as a connection on the infinite-dimensional vector bundle $C^\infty(M)[[\hbar]] \times B \rightarrow B$, makes the family of *-products parallel;

$$d\tau^1 + [\tau^1, \tau^1]/2 + [\tau^0 + m, \tau^2] = 0,$$

i.e. the curvature of τ^1 is an inner derivation of the algebra $C^\infty(M)[[\hbar]]$ with its *-product $\tau^0 + m$, and finally

$$d\tau^2 + [\tau^1, \tau^2] = 0.$$

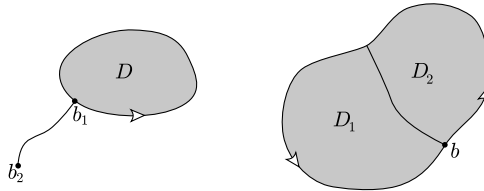
For any point $b \in B$, let us denote the algebra $C^\infty(M)[[\hbar]]$ with the *-product $m + \tau^0(b)$ by A_b .

Although τ^1 is a connection on an infinite-dimensional vector bundle, its parallel transport is well-defined (this is because $\tau^1 \in \Omega^1(B) \otimes \hbar D(M)[[\hbar]]$, i.e. $\tau^1 = O(\hbar)$). One can also use τ^2 to get a “2-dimensional parallel transport”, and we get the following:

- (1) for every curve γ in B , connecting points b_1 and b_2 , an isomorphism $T_\gamma : A_{b_1} \rightarrow A_{b_2}$; T_γ is just the parallel transport along γ ,
- (2) for every disk D in B with a chosen point b on the boundary, an element $a_{D,b} \in A_b$.

The following relations then hold:

- (1) if γ is the boundary of D (so that $b_1 = b_2 = b$) then T_γ is the inner automorphism given by $a_{D,b} \in A_b$
- (2) $a_{D,b} \in A_b$ depends only on the homotopy class of D rel boundary
- (3) $a_{D,b}$'s are multiplicative ($a_{D_1 \cup D_2, b} = a_{D_1, b} a_{D_2, b}$) and behave naturally under change of b ($a_{D, b_2} = T_\gamma a_{D, b_1}$ where γ is the curve from b_1 to b_2), see the picture:



The quantization of Hamiltonian families to tight *-product families, as we have just described it, depends on the choice of the quasiisomorphism $L' \rightarrow L''$. Unfortunately, there is no natural (i.e. diffeomorphisms-invariant) choice for the quasiisomorphism, but fortunately, it is natural up

to homotopy. Using the precise meaning of the previous sentence, given by Kontsevich in [3], we have:

Proposition 4.1. *Given a Hamiltonian family of formal Poisson structures on M parametrized by B , and given also a smooth family of connections on M parametrized by B , there is a natural and local construction of a tight family of $*$ -products on M parametrized by B .*

5. DEFORMATION QUANTIZATION OF DIRAC STRUCTURES

Let now N be a manifold and $C(0)$ a regular Dirac structure on N . It means that we are given a foliation F of N and a closed 2-form on the leaves of F . Suppose we extend $C(0)$ to a formal family $C(\hbar)$ of Dirac structures.

Let us choose a connection on the vector bundle TN/F (the normal bundle of the foliation), to be able to use Proposition 4.1. If $M \subset N$ is a local transversal, i.e. a submanifold with complementary dimension to F and transversal to the leaves, the connection induces a connection on TM . The Dirac structure $C(\hbar)$ pulls back to a Poisson structure on M of the form

$$\pi = \pi^{(1)}\hbar + \pi^{(2)}\hbar^2 + \pi^{(3)}\hbar^3 + \dots$$

More generally, for any smooth map $f : M \times B \rightarrow N$ such that

- ▷ for any point $p = (m, b) \in M \times B$, $d_p f$ maps $T_b B$ into $F_{f(p)}$, and the induced map $T_m M \rightarrow (T_{f(p)} N) / F_{f(p)}$ is a bijection,

the pullback of $C(\hbar)$ is a formal family of Dirac structures on $M \times B$ satisfying the conditions of Lemma 3.2. Together with the pullback of the connection, it gives us (via Proposition 4.1) a tight $*$ -product family on M parametrized by B , and thus the isomorphisms T and elements a as in Section 4.

From here it is not difficult to see that the quantizations of the Poisson structures on transversals form a stack on N . One needs the cases $B = I$ (I is an interval), i.e. homotopies of transversals, $B = I \times I$, i.e. homotopies of homotopies, and finally $B = I \times I \times I$. We shall construct the stack in some detail in what follows.

5.1. Algebroids. A *linear category* is a category in which $\text{Hom}(X, Y)$ is a vector space (for any two objects X, Y) and composition of morphisms is bilinear. An *algebroid* is a linear category in which any two objects are isomorphic. Any algebroid gives us an algebra given up to isomorphisms, namely the algebra $\text{Hom}(X, X)$ for whatever object X . We shall quantize $C(\hbar)$ into a stack of linear categories, by first describing a prestack of algebroids and then stackifying.

5.2. The case of contractible foliations. Let us suppose that the foliation (N, F) is isomorphic to a fibration of the type $M_0 \times \mathbb{R}^k \rightarrow M_0$ for some M_0 and k . The point is that every foliation is locally of this form. We shall construct an algebroid $A(N)$. The objects of this category are the cross-sections³ of the foliation. Morphisms are defined as follows (this construction, together with the very definition of algebroids, is taken from [3]): For any object X , the algebra $\text{Hom}(X, X)$ is the quantized algebra of functions on X . To define the space $\text{Hom}(X, Y)$, which should be a $\text{Hom}(X, X)$ - $\text{Hom}(Y, Y)$ bimodule, we first choose a smooth homotopy that moves X to Y , while every point stays in its leaf. This homotopy gives us an isomorphism between the algebras $\text{Hom}(X, X)$ and $\text{Hom}(Y, Y)$, and we define $\text{Hom}(X, Y)$ to be the diagonal bimodule (i.e. the graph of the isomorphism). If we choose a different homotopy from X to Y , we have to identify the two definitions of $\text{Hom}(X, Y)$. To do it, we choose a homotopy between these homotopies, which gives us an element $a \in \text{Hom}(X, X)$ (see Section 4), and we identify the two $\text{Hom}(X, Y)$'s by multiplication by a . The element a doesn't depend on the choice of homotopy of homotopies, since any two such choices are homotopic.

³i.e. submanifolds of N intersecting each leaf transversally and once (in our case they can be, of course, identified with maps $M_0 \rightarrow \mathbb{R}^k$)

5.3. Construction of the stack. Let us return to the case of a general (N, F) . For any open subset $U \subset N$, on which the foliation satisfies the condition of the previous subsection, we get the algebroid $A(U)$. Open sets of this kind form a category Op_{res} : there is a (unique) morphism $V \rightarrow U$ between two such sets iff $V \subset U$. We first define a linear category A fibred over Op_{res} , whose fibres are $A(U)$'s: whenever $V \subset U$, for any object $X \in A(V)$ (i.e. any cross-section of V) and $Y \in A(U)$, we define $\text{Hom}(X, Y)$ by choosing a homotopy in U that moves X along the leaves to a (uniquely defined) open subset \bar{Y} of Y , and then continuing as in the previous subsection, i.e. setting $\text{Hom}(X, Y)$ to be the graph of the isomorphism between the quantized algebras of functions on X and \bar{Y} .

The rest is purely formal. First we extend A to a linear category A' fibred over Op (the category of all open sets in N). This is done by induction (right adjoint to the restriction). Recall from [2] its possible construction: Any object $U' \in Op$ gives a full subcategory $Op_{res}(U')$ of Op_{res} , of objects contained in U' . One then defines $A'(U') = \text{Hom}_{/Op_{res}}(Op_{res}(U'), A)$. We thus get a prestack over N (a category fibred over Op), which we finally stackify.

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